Cosmological Scaling Solutions of Multiple Tachyon Fields with Inverse Square Potentials

Zong-Kuan Guo[*b] and Yuan-Zhong Zhang[a,b]

[a]*CCAST (World Lab.), P.O. Box 8730, Beijing 100080*
[b]*Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100080, China*

Abstract

We investigate cosmological dynamics of multiple tachyon fields with inverse square potentials. A phase-space analysis of the spatially flat FRW models shows that there exists power-law cosmological scaling solutions. We study the stability of the solutions and find that the potential-kinetic-scaling solution is a global attractor. However, in the presence of a barotropic fluid the solution is an attractor only in one region of the parameter space and the tracking solution is an attractor in the other region. We briefly discuss the physical consequences of these results.

[*]e-mail address: guozk@itp.ac.cn

1 Introduction

Cosmological inflation has become an integral part of the standard model of the universe. Apart from being capable of removing the shortcomings of the standard cosmology, the paradigm seems to have gained a fairly good amount of support from the recent observations on microwave background radiation. On the other hand, there have been difficulties in obtaining accelerated expansion from fundamental theories such as string/M-theory. Recently, Sen [1] has constructed a classical time-dependent solution which describes the decay process of an unstable D-brane in the open string theory. During the decay process the tachyon field on the brane rolls down toward the minimum of the potential. The tachyon field might be responsible for cosmological inflation at the early epochs due to tachyon condensation near the top of the effective potential [2], and could contribute to some new form of cosmological dark matter at late times. Several authors have investigated the process of rolling of the tachyon in the cosmological background [3] and in the braneworld scenario [4].

The stability of tachyon inflation against changes in initial conditions has been studied for exponential potentials [5] and for inverse power-law potentials [6]. Exact solutions for a purely tachyon field with an inverse square potential are known [7], but no solutions exist for multiple tachyon fields, so a dynamical analysis may be relevant. It is interesting that the inverse square potentials play the same role for tachyon fields as the exponential potentials do for standard scalar fields [8]. The potentials allow constructing an autonomous system which gives power-law solutions. Recently, it was pointed out that the tachyon model of inflation is hard to be consistent with observations [9], unless there is a large number of coincident branes [10]. In this letter, we will consider cosmological dynamics of multiple tachyon fields with inverse square potentials. We have assumed that there is no direct coupling between the inverse square potentials. The only interaction is gravitational. A phase-space analysis of the spatially flat FRW models shows that there exist a cosmological scaling solution which is a unique attractor. We find that accelerated expansion of the universe can be driven by multiple tachyon fields at lower-Planck energy densities. In this model we then introduce a barotropic perfect fluid to the system. We discuss the physical consequences of these results.

2 Multiple Inverse Square Potentials

We start with more general model with m tachyon fields ϕ_i, in which each has an inverse square potential

$$V_i(\phi_i) = V_{0i}\phi_i^{-2}. \tag{1}$$

Note that there is no direct coupling of the tachyon fields, which influence each other only via their effect on the expansion. The evolution equation of each tachyon field for a spatially

flat FRW model with Hubble parameter H is

$$\frac{\ddot{\phi}_i}{1 - \dot{\phi}_i^2} + 3H\dot{\phi}_i + \frac{1}{V_i(\phi_i)}\frac{dV_i(\phi_i)}{d\phi_i} = 0, \tag{2}$$

subject to the Friedmann constraint

$$H^2 = \frac{\kappa^2}{3}\sum_{i=1}^{m}\frac{V_i(\phi_i)}{\sqrt{1 - \dot{\phi}_i^2}}, \tag{3}$$

where $\kappa^2 \equiv 8\pi G_N$ is the gravitational coupling. Defining $2m$ dimensionless variables

$$x_i \equiv \dot{\phi}_i, \tag{4}$$

$$y_i \equiv \frac{\kappa^2 V_i(\phi_i)}{3H^2}, \tag{5}$$

the evolution equations (2) can be written as an autonomous system:

$$x_i' = -3\left(x_i - \sqrt{\beta_i y_i}\right)\left(1 - x_i^2\right), \tag{6}$$

$$y_i' = 3y_i\left(\sum_{i=1}^{m}\frac{y_i x_i^2}{\sqrt{1 - x_i^2}} - \sqrt{\beta_i y_i}\, x_i\right), \tag{7}$$

where a prime denotes a derivative with respect to the logarithm of the scalar factor, $N \equiv \ln a$, and $\beta_i \equiv 4/(3\kappa^2 V_{0i})$. The constraint equation (3) becomes

$$1 = \sum_{i=1}^{m}\frac{y_i}{\sqrt{1 - x_i^2}}. \tag{8}$$

Critical points correspond to fixed points where $x_i' = 0$ and $y_i' = 0$, and there are self-similar solutions with

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\sum_{i=1}^{m}\frac{y_i x_i^2}{\sqrt{1 - x_i^2}}. \tag{9}$$

This corresponds to an expanding universe with a scale factor $a(t)$ given by $a \propto t^p$ or a contracting universe with a scalar factor given by $a \propto (-t)^p$, where

$$p = \frac{1}{\frac{3}{2}\sum_{i=1}^{m}\frac{y_i x_i^2}{\sqrt{1 - x_i^2}}}. \tag{10}$$

Setting $x_i' = 0$ and $y_i' = 0$, we can get the fixed points K corresponding to kinetic-dominated solutions and the fixed points S listed in Table 1 with

$$x_s^2 = \frac{1}{2}\beta\left(\sqrt{\beta^2 + 4} - \beta\right), \tag{11}$$

2

Label	x_i	y_i	p	Existence	Stability
K	± 1	0	$\frac{2}{3}$	all β_i	unstable
S	x_s	$\frac{x_s^2}{\beta_i}$	$\frac{4}{3\beta(\sqrt{\beta^2+4}-\beta)}$	all β_i	stable

Table 1: The properties of the critical points in a spatially flat FRW universe containing two tachyon fields with inverse square potentials.

where we using the following definition:

$$\frac{1}{\beta} \equiv \sum_{i=1}^{m} \frac{1}{\beta_i}. \tag{12}$$

The fixed points K behave like non-relativistic matter with $p = 2/3$. The fixed point S is the kinetic-potential-scaling solution with

$$p = \frac{4}{3\beta(\sqrt{\beta^2+4}-\beta)}. \tag{13}$$

We find that accelerated expansion of the universe occurs if $\beta < 2/\sqrt{3}$. For a single tachyon field with an inverse square potential, V_0 must be larger than $2/(\sqrt{3}\kappa^2)$ to guarantee accelerated expansion. However, accelerated expansion of the universe can be driven by sufficiently multiple tachyon fields with V_0 at lower-Planck energy densities.

In order to analysis the stability of the critical points, we only consider the cosmologies containing two tachyon fields. Using the Friedmann constraint equation (8), we reduce Eqs.(6) and (7) to three independent equations

$$x_1' = -3\left(x_1 - \sqrt{\beta_1}y_1\right)\left(1 - x_1^2\right), \tag{14}$$

$$y_1' = 3y_1\left[\frac{(x_1^2 - x_2^2)y_1}{\sqrt{1-x_1^2}} + x_2^2 - \sqrt{\beta_1 y_1}\, x_1\right], \tag{15}$$

$$x_2' = -3\left[x_2 - \sqrt{\beta_2}\left(1 - x_2^2\right)^{\frac{1}{4}}\left(1 - \frac{y_1}{\sqrt{1-x_1^2}}\right)^{\frac{1}{2}}\right]\left(1 - x_2^2\right). \tag{16}$$

The linearization of system (14)-(16) about the fixed points K yields three positive characteristic exponents

$$\lambda_1 = \lambda_2 = 6,\ \lambda_3 = 3,$$

which indicate that the kinetic-dominated solutions are always unstable. Substituting linear perturbations about the critical point S into Eqs.(14)-(16), to first-order in the perturbations, yields the three negative eigenvalues

$$\lambda_1 = -\frac{3x_s^2}{2\beta^2}(2x_s^2 + \beta^2),$$

3

$$\lambda_2 = -\frac{3x_s^2}{4\beta^2}\left[(2x_s^2 + \beta^2) + \sqrt{(2x_s^2 + \beta^2)^2 - 16\beta^2 x_s^2(1 + \beta_2/\beta_1)(1 - \beta/\beta_1)}\right],$$

$$\lambda_3 = -\frac{3x_s^2}{4\beta^2}\left[(2x_s^2 + \beta^2) - \sqrt{(2x_s^2 + \beta^2)^2 - 16\beta^2 x_s^2(1 + \beta_2/\beta_1)(1 - \beta/\beta_1)}\right].$$

The critical point is consequently stable so that the corresponding cosmological scaling solution is always a global attractor for any β. For a single tachyon field model $\beta = \beta_1$, the three eigenvalues reduce to one eigenvalue.

3   Plus a Barotropic Perfect Fluid

We now consider multiple tachyon fields with inverse square potentials evolving in a spatially flat FRW universe containing a fluid with barotropic perfect equation of state $P_\gamma = (\gamma-1)\rho_\gamma$, where γ is a constant, $0 < \gamma \leq 2$, such as radiation ($\gamma = 4/3$) or dust ($\gamma = 1$). The evolution equation for the barotropic perfect fluid is

$$\dot{\rho}_\gamma = -3H(\rho_\gamma + P_\gamma), \tag{17}$$

subject to the Fridemann constraint

$$H^2 = \frac{\kappa^2}{3}\left(\sum_{i=1}^{m}\frac{V_i(\phi_i)}{\sqrt{1 - \dot{\phi}_i^2}} + \rho_\gamma\right). \tag{18}$$

We define another dimensionless variable $z \equiv \frac{\kappa^2 \rho_\gamma}{3H^2}$. The evolution equations (2) and (17) can then be written as an autonomous system:

$$x_i' = -3\left(x_i - \sqrt{\beta_i y_i}\right)\left(1 - x_i^2\right), \tag{19}$$

$$y_i' = 3y_i\left(\sum_{i=1}^{m}\frac{y_i x_i^2}{\sqrt{1 - x_i^2}} - \sqrt{\beta_i y_i}\,x_i + \gamma z\right), \tag{20}$$

$$z' = 3z\left(\sum_{i=1}^{m}\frac{y_i x_i^2}{\sqrt{1 - x_i^2}} + \gamma z - \gamma\right), \tag{21}$$

and the Fridemann constraint equation becomes

$$1 = \sum_{i=1}^{m}\frac{y_i}{\sqrt{1 - x_i^2}} + z. \tag{22}$$

Critical points correspond to fixed points where $x_i' = 0$, $y' = 0$ and $z' = 0$, and there are self-similar solutions with

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\left(\sum_{i=1}^{m}\frac{y_i x_i^2}{\sqrt{1 - x_i^2}} + \gamma z\right). \tag{23}$$

4

Label	x_i	y_i	z	p	Existence	Stability
K	± 1	0	0	$\frac{2}{3}$	all β_i, γ	unstable
S	x_s	$\frac{x_s^2}{\beta_i}$	0	$\frac{4}{3\beta(\sqrt{\beta^2+4}-\beta)}$	all β_i, γ	stable ($\gamma \geq \frac{\beta\sqrt{\beta^2+4}-\beta^2}{2}$); unstable ($\gamma < \frac{\beta\sqrt{\beta^2+4}-\beta^2}{2}$)
F	0	0	1	$\frac{2}{3\gamma}$	all β_i, γ	stable ($\gamma = 0$); unstable ($\gamma \neq 0$)
T	$\sqrt{\gamma}$	$\frac{\gamma}{\beta_i}$	$\frac{\beta\sqrt{1-\gamma}-\gamma}{\beta\sqrt{1-\gamma}}$	$\frac{2}{3\gamma}$	$\gamma < \frac{\beta\sqrt{\beta^2+4}-\beta^2}{2}$	stable

Table 2: The properties of the critical points in a spatially flat FRW universe containing two tachyon fields with inverse square potentials plus a barotropic perfect fluid.

This corresponds to an expanding universe with a scale factor $a(t)$ given by $a \propto t^p$, where

$$p = \frac{2}{3 \sum_{i=1}^m \frac{y_i x_i^2}{\sqrt{1-x_i^2}} + 3\gamma z}. \tag{24}$$

The system (19)-(21) has a fixed point K corresponding to kinetic-dominated solution, a fixed point S which is a kinetic-potential-scaling solution, a fixed point F which is a fluid-dominated solution, and a fixed point T which is a tracking solution listed in Table 2. In order to analysis the stability of the critical points, we still consider the cosmologies containing two tachyon fields plus a barotropic perfect fluid.

K: $x_i = \pm 1$, $y = 0$, $z = 0$. These kinetic-dominated solutions always exist for all β_i and γ, which behave like non-relativistic matter with $a \propto t^{2/3}$ irrespective of the nature of the potentials. The linearization of system (19)-(21) about these fixed points yields four eigenvalues

$$\lambda_1 = \lambda_2 = 6, \ \lambda_3 = \lambda_4 = 3\gamma,$$

which indicate that the solutions are always unstable.

S: $x_i = x_s$, $y = x_s^2/\beta_i$, $z = 0$. The potential-kinetic-scaling solution exists for all β_i and γ. The power-law exponent (13) depends on the parameter β of the potentials. The linearization of system (19)-(21) about the fixed point yields four eigenvalues

$$
\begin{aligned}
\lambda_1 &= -3\left(\frac{x_s^4}{\beta^2} + \frac{x_s^2}{2}\right), \\
\lambda_2 &= -3(\gamma - x_s^2), \\
\lambda_3 &= -\frac{3}{2}\left(\frac{x_s^4}{\beta^2} + \frac{x_s^2}{2}\right) + \frac{3}{2}\sqrt{\left(\frac{x_s^4}{\beta^2} + \frac{x_s^2}{2}\right)^2 - \frac{4x_s^6}{\beta^2}}, \\
\lambda_4 &= -\frac{3}{2}\left(\frac{x_s^4}{\beta^2} + \frac{x_s^2}{2}\right) - \frac{3}{2}\sqrt{\left(\frac{x_s^4}{\beta^2} + \frac{x_s^2}{2}\right)^2 - \frac{4x_s^6}{\beta^2}},
\end{aligned}
$$

5

which indicate that the solutions are unstable for $\gamma < \beta(\sqrt{\beta^2 + 4} - \beta)/2$ and stable for $\gamma \geq \beta(\sqrt{\beta^2 + 4} - \beta)/2$ from Eq.(11).

F: $x_i = 0$, $y_i = 0$, $z = 1$. The fluid-dominated solution exists for all β_i and γ, corresponding to a power-law solution with $p = 2/3\gamma$. The linearization of system (19)-(21) about the fixed point yields four eigenvalues

$$\lambda_1 = \lambda_2 = -3, \ \lambda_3 = \lambda_4 = 3\gamma,$$

which indicate that the solution is unstable for $\gamma \neq 0$ and stable only for $\gamma = 0$.

T: $x_i = \sqrt{\gamma}$, $y_i = \gamma/\beta_i$, $z = 1 - \frac{\gamma}{\beta\sqrt{1-\gamma}}$. The tracking solution exists for $\gamma < \beta(\sqrt{\beta^2 + 4} - \beta)/2$. The solution displays a tracking behavior according to the definition in [11]. The power-law exponent, $p = 2/3\gamma$, is identical to that of the fluid-dominated solution, depends only on the barotropic index γ and is independent of the parameters β_i of the potentials. The linearization of system (19)-(21) about the fixed point yields four eigenvalues

$$
\begin{aligned}
\lambda_1 &= -\frac{3}{4}(2 - \gamma) + \frac{3}{4}\sqrt{4 - 20\gamma + 17\gamma^2}, \\
\lambda_2 &= -\frac{3}{4}(2 - \gamma) - \frac{3}{4}\sqrt{4 - 20\gamma + 17\gamma^2}, \\
\lambda_3 &= -\frac{3}{4}(2 - \gamma) + \frac{3}{4}\sqrt{(2 - \gamma)^2 + 16\gamma\sqrt{1 - \gamma}\left(\gamma/\beta - \sqrt{1 - \gamma}\right)}, \\
\lambda_4 &= -\frac{3}{4}(2 - \gamma) - \frac{3}{4}\sqrt{(2 - \gamma)^2 + 16\gamma\sqrt{1 - \gamma}\left(\gamma/\beta - \sqrt{1 - \gamma}\right)},
\end{aligned}
$$

which indicate that the solution is always stable when this point exists for $\gamma < \beta(\sqrt{\beta^2 + 4} - \beta)/2$.

The different regions in the (β, γ) parameter space lead to different qualitative evolution in Figure 1. For $\gamma > \beta(\sqrt{\beta^2 + 4} - \beta)/2$, K, S and F exist. Point S is the stable late-time attractor. Hence generic solutions begin at the two kinetic-dominated solutions or at the fluid-dominated solution and approach the kinetic-potential-scaling solution at late times. For $\gamma < \beta(\sqrt{\beta^2 + 4} - \beta)/2$, all critical points exist. Point T is the stable late-time attractor. Hence generic solutions start at the two kinetic-dominated solution, at the kinetic-potential-scaling solution or at the fluid-dominated solution and approach the stable fluid-kinetic-potential-scaling solution.

4   Conclusions and Discussions

We have presented a phase-space analysis of the evolution for a spatially flat FRW universe containing m tachyon fields with inverse square potentials. We find that there exist
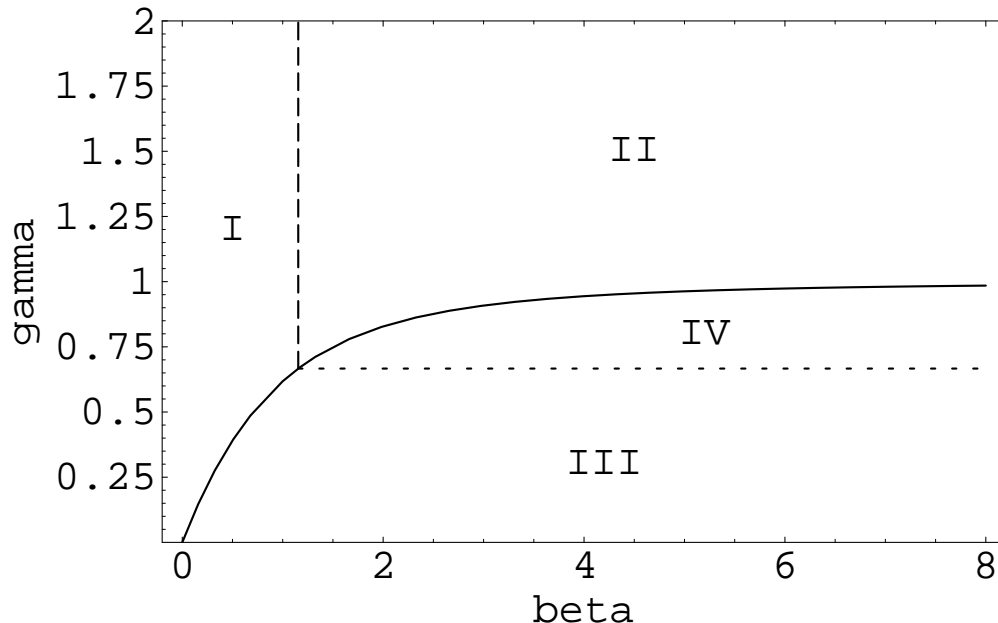
Figure 1: Stability regions of the (β, γ) parameter space. In the regions I and II, the tachyon kinetic-potential scaling solution is the stable late-time attractor. In the regions III and IV, the fluid-dominated solution is the stable late-time attractor. The universe accelerates in the regions I and III, while the universe decelerates in the regions II and IV.

cosmological scaling solutions that corresponds to an expanding universe with $a \propto t^p$. For a single tachyon field with an inverse square potential, the universe could accelerate only at nearly Planckian energy densities. However, accelerated expansion of the universe can be driven by sufficiently multiple tachyon fields even at lower-Planck energy densities. The reason for this behavior is that while each field experiences the 'downhill' force from its own potential, it feels the friction from all the tachyon fields via their contribution to the expansion [12]. In order to analysis the stability of the critical points, we only consider the cosmologies containing two tachyon fields. We find that the critical point is always stable so that the scaling solution is a global attractor irrespective of the form of the potentials. This implies that the velocity of each tachyon field tends to be equal and constant via their effect on the expansion.

Then we have extended the phase-space analysis of the evolution to a realistic universe model with a barotropic perfect fluid plus m tachyon fields with inverse square potentials. We have shown that the energy density of the tachyon dominates at late times for $\gamma > \beta(\sqrt{\beta^2 + 4} - \beta)/2$, In constraint, for $\gamma < \beta(\sqrt{\beta^2 + 4} - \beta)/2$, the barotropic fluid does not dominate completely and the contribution of tachyon energy density to the total one is not negligible.

We emphasize that we have assumed that there is no direct coupling between these

inverse square potentials. In a system with m non-coincident but parallel non-BPS D3-branes [13], there are two kinds of open strings. One of them starts from and ends on the same brane; The other starts from a given brane, then ends on a different brane. If the distance between two branes are much larger than the string scale, one can ignore the second kind of open strings, leaving a tachyon on the world volume for every brane. Thus, we have m tachyons without interaction and the action is simply the sum of m single-tachyon actions. It is worth studying further the the cosmological dynamics of multiple tachyon fields with interactions.

Acknowledgements

References

[1] A.Sen, JHEP **0204** (2002) 048;
A.Sen, JHEP **0207** (2002) 065.

[2] G.W.Gibbons, Phys.Lett. **B537** (2002) 1;
M.Fairbairn and M.H.Tytgat, Phys.Lett. **B546** (2002) 1;
G.Shiu and I.Wasserman, Phys.Lett. **B541** (2002) 6.

[3] M.Sami, P.Chingangbam and T.Qureshi, Phys.Rev. **D66** (2002) 043530;
Y.S.Piao, Q.G.Huang, X.M.Zhang and Y.Z.Zhang, Phys.Lett. **B570** (2003) 1;
J.Kluson, JHEP **0401** (2004) 019;
M.Majumdar and A.C.Davis, hep-th/0304226.

[4] M.C.Bento, O.Bertolami and A.A.Sen, Phys.Rev. **D67** (2003) 063511;
D.Bazeia, F.A.Brito and J.R.Nascimento, Phys.Rev. **D68** (2003) 085007;
M.Sami, Mod.Phys.Lett. **A18** (2003) 691;
B.C.Paul and M.Sami, hep-th/0312081;
G.Calcagni, hep-ph/0402126.

[5] Z.K.Guo, Y.S.Piao, R.G.Cai and Y.Z.Zhang, Phys.Rev. **D68** (2003) 043508;
Z.K.Guo, H.S.Zhang and Y.Z.Zhang, Phys.Rev. **D69** (2004) 063502.

[6] L.R.W.Abramo and F.Finelli, Phys. Lett. **B575** (2003) 165.

[7] T.Padmanabhan, Phys.Rev. **D66** (2002) 021301.

[8] I.P.C.Heard and D.Wands, Class.Quant.Grav. **19** (2002) 5435;
Z.K.Guo, Y.S.Piao and Y.Z.Zhang, Phys.Lett. **B568** (2003) 1;
Z.K.Guo, Y.S.Piao, R.G.Cai and Y.Z.Zhang, Phys.Lett. **B576** (2003) 12;
J.M.Aguirregabiria and R.Lazkoz, hep-th/0402190.

[9] L.Kofman and A.Linde, JHEP **0207** (2002) 004.

[10] Y.S.Piao, R.G.Cai, X.M.Zhang and Y.Z.Zhang, Phys.Rev. **D66** (2002) 121301.

[11] A.R.Liddle and R.J.Scherrer, Phys.Rev. **D59** (1999) 023509.

[12] A.R.Liddle, A.Mazumdar and F.E.Schunck, Phys.Rev. **D58** (1998) 061301;
Y.S.Piao, W.Lin, X.M.Zhang and Y.Z.Zhang, Phys.Lett. **B528** (2002) 188.

[13] A.Mazumdar, S.Panda and A.Perez-Lorenzana, Nucl.Phys. **B614** (2001) 101;
N.T.Jones and S.H.H.Tye, JHEP **0301** (2003) 012;
N.T.Jones and S.H.H.Tye, JHEP **0308** (2003) 037.